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The Boltzmann equation for gluons at early times after a heavy ion collision

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Abstract

A Boltzmann equation is given for the early stages of evolution of the gluon system produced in a head-on heavy ion collision. The collision term is taken from gluon-gluon scattering in the one-gluon approximation. $< p_\perp >$ and $< p_z^2 >$ are evaluated as a function of time using initial conditions taken from the McLerran-Venugopalan model.

1 Introduction

In an earlier paper[1] we studied the early stages of the gluon system produced in a heavy ion collision. In that paper we took the initial conditions of the gluon system from the McLerran-Venugopalan model assuming that all gluons at or below the saturation momentum in the light-cone wavefunction of the McLerran-Venugopalan model are freed in the heavy ion collision while all gluons beyond the saturation momentum are not freed[2, 3, 4, 5, 6]. We then wrote an equation for the rate of change of the *typical* transverse momentum of a gluon in the system as a function of time. This equation takes into account, in an average way, the transfer of transverse momentum into longitudinal momentum due to elastic gluon-gluon scattering in the one gluon exchange approximation. We were able to follow the decrease of the typical transverse momentum of a gluon with time up to times on the order

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of $\frac{1}{Q_s} exp[const/\sqrt{\alpha}\ ]$ at which time the typical transverse and longitudinal momenta become equal, and our procedure broke down. We argued that this was likely the time at which equilibrium was setting in and our simple approach was not capable of following the gluon system into equilibrium.

In this note we reformulate our previous discussion in a much more general way in terms of the Boltzman equation[7] with the collision term taken from elastic gluon-gluon scattering in the one gluon exchange approximation. In the logarithmic approximation for integrating over the angle of scattering, the same approximation used in Ref.1, the Boltzmann equation reduces to a Fokker-Planck equation[7], given in (15) and (16), for the number of gluons, f, per unit of phase space. The infrared cutoff on very small angle scattering is taken from Ref.1 although the exact form of this cutoff is not crucial for the existence of an equation of the type given in (15).

There are two explicit parameters appearing in (15). The first of these parameters, η, which is time-independent is given by the initial conditions for the evolution. In our explicit evaluations we take η from the McLerran-Venugopalan model in terms of the saturation momentum Q_s. The second parameter, η_{-1}, given in (13) should be determined self-consistently from the solution to (15). In our explicit evaluations we take η_{-1} to be given by the McLerran-Venugopalan model at very early times (28), while the time dependence of η_{-1} can be estimated using (32). We have also assumed t_o, in (10), to be time-independent. In Ref.1, we found the time-dependence of t_o to have a negligible effect at early times and we expect the same to be true here. However, before attempting to solve (15) more completely, say by numerical means, at non early times one must give a specification of t_o, in (10), analogous to what was done in Ref.1, but now in the context of the Boltzmann equation.

2 The Boltzmann equation

A derivation of the Boltzmann equation can be found in Ref.7, and can be written as

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \bigtriangledown_{\vec{x}} f = C \qquad (1)$$

where C is the collision term. f is the number of gluons per unit of phase

space after the valence quarks have separated in a high energy, head-on, heavy ion collision. Thus $f = f(\vec{p}, \vec{x}, t)$ and

$$N(\vec{x}, t) = \int d^3\vec{p} f \tag{2}$$

is the gluon density per unit volume. f is a scalar function. We shall assume, for simplicity, that f only depends on z, the coordinate corresponding to the axis along which the heavy ions collide, but not on the transverse coordinates $\underline{x}$. Further we suppose that the physics of the heavy ion collision is boost invariant, at least for final state particles not having too large a rapidity in the center of mass system. If we parameterize the momentum and position of a final state gluon by

$$p_\mu = p_\perp(cosh\ y, \hat{\underline{\mathrm{v}}}, sinh\ y) = (p_0, \underline{p}, p_z) \tag{3}$$

$$x_\mu = \tau(cosh\ \eta, \underline{0}, sinh\ \eta) = (t, \underline{\mathrm{x}}, z) \tag{4}$$

then $f = f(p_\perp, \tau, y - \eta)$ expresses the boost invariance of the distribution of particles in the final state. In terms of τ, y, η

$$(\frac{\partial}{\partial t} + \vec{v} \cdot \bigtriangledown_{\vec{x}})f = (\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z})f = \frac{cosh(y-\eta)}{coshy}(\frac{\partial}{\partial t} - \frac{tanh(y-\eta)}{\tau}\frac{\partial}{\partial y})f \tag{5}$$

or, taking $\eta = z = 0$

$$(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z})f = (\frac{\partial}{\partial t} - \frac{p_z}{t}\frac{\partial}{\partial p_z})f(t, p_\perp, p_z) \tag{6}$$

where we have expressed, at $z = 0$, f as a function of $t, p_\perp$ and p_z, variables convenient for the discussion to follow.

Now turn to the collision term in (1). It is straightforward to show that

$$C = \partial_i \int d^3p_2[(\partial_j - \partial_{2j})ff_2]B_{ij} \tag{7}$$

where we use the notation $\partial_i = \frac{\partial}{\partial p_i}, \partial_{2i} = \frac{\partial}{\partial p_{2i}}, f = f(t, p_\perp, p_z), f_2 = f(t, p_{2\perp}, p_{2z})$. B_{ij} is given by

$$B_{ij} = 2\pi\alpha^2(\frac{N_c^2}{N_c^2 - 1})L[(1 - \vec{v} \cdot \vec{v}_2)\delta_{ij} + v_i v_{2j} + v_{2i} v_j)] \tag{8}$$

with $\vec{v} = \frac{\vec{p}}{|\vec{p}|}$ and $\vec{v}_2 = \frac{\vec{p}_2}{|\vec{p}_2|}$. Except for the factor $(\frac{N_c^2}{N_c^2-1})$ the B_{ij} in (8) is identical to that given in Ref.7 for QED. The factor L represents a (perturbatively divergent) integral over the (small) angle of scattering which we cut off as in Ref.1 at a minimum scattering angle θ_m given by

$$\theta_m^3 = \frac{t_0}{t} \tag{9}$$

with t_0 slowly varying with time for t not too large. We shall treat t_0 as a constant in this paper. Thus,

$$L = \int_{\theta_m}^{1} \frac{d\theta}{\theta} = \frac{1}{3}\ell n\ t/t_0 \equiv \frac{1}{3}\xi. \tag{10}$$

Using (8) and (10) in (7) along with (1) and (6) leads to

$$C = \lambda N\xi \bigtriangledown_{\vec{p}}^2 f((t,\vec{p}) + 2\lambda N_{-1}\xi \bigtriangledown_{\vec{p}} \cdot(\vec{v}f(t,\vec{p})) \tag{11}$$

where

$$N(t) \int d^3p f(t,\vec{p}) \tag{12}$$

is the gluon number density while

$$N_\gamma(t) = \int d^3p f(t,\vec{p})|\vec{p}|^\gamma \tag{13}$$

and where

$$\lambda = \frac{2\pi\alpha^2}{3}(\frac{N_c^2}{N_c^2-1}). \tag{14}$$

Defining $tN = \eta, tN_{-1} = \eta_{-1}$ and using (6) and (11) in (1) finally leads to an equation of the Fokker-Planck type

$$(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial p_z}p_z)\tilde{f} = \lambda\eta\xi \bigtriangledown_{\vec{p}}^2 \tilde{f} + 2\lambda\eta_{-1}\xi \bigtriangledown_{\vec{p}} (\vec{v}\tilde{f}) \tag{15}$$

with

$$tf(t,\vec{p}) = \tilde{f}(t,\vec{p}). \tag{16}$$

3 Properties of the solution of the Boltzmann equation

It appears difficult to solve (15) even with simple initial conditions. Nevertheless, one can discern general characteristics of the solutions, for early times, even without being able to completely solve the equation. Taking the integral with respect to d^3p on both sides of (15) and recalling that $\int \tilde{f} d^3p = \eta$ one finds

$$\frac{\partial}{\partial \xi}\eta = 0 \tag{17}$$

which means that the number density of gluons $N(t)$ decreases like $1/t$.

Multiplying both sides of (15) by $p_\perp^2$ and integrating over all $\vec{p}$ leads to

$$\frac{\partial}{\partial \xi} < p_\perp^2 >= 4\lambda\eta\xi(1 - \frac{\eta_{-1}\eta_{+1}}{\eta^2}) + \frac{4\lambda\eta_{-1}\xi}{\eta}\int d^3p\tilde{f}\frac{p_z^2}{p} \tag{18}$$

while multiplication by p_z^2 and integration yields

$$\frac{\partial}{\partial \xi} < p_z^2 > +2 < p_z^2 >= 2\lambda\eta\xi - \frac{4\lambda\eta_{-1}\xi}{\eta}\int d^3p\tilde{f}\frac{p_z^2}{p}. \tag{19}$$

In (18) and (19) we have used

$$\int d^3p\tilde{f}p_\perp^2 \equiv < p_\perp^2 > \eta \tag{20}$$

and

$$\int d^3p\tilde{f}p_z^2 \equiv < p_z^2 > \eta. \tag{21}$$

In case $\tilde{f}$ does not decrease faster than $p_\perp^{-4}$, which is the behavior in perturbative QCD, it may be useful to replace (18) by

$$\frac{\partial}{\partial \xi} < p_\perp >= \lambda\xi(\eta_{-1}^{\perp} - \frac{2\eta_{-1}\eta(p_\perp/p)}{\eta}) \tag{22}$$

where

$$\eta_{-1}^{\perp} = \int d^3p\tilde{f}\frac{1}{p_\perp} \tag{23}$$

and

$$\eta(p_\perp/p) = \int d^3p \tilde{f} \frac{p_\perp}{p}. \quad (24)$$

At early times the values of p_z are very small at $z = 0$ so that we may set $\eta^\perp_{-1} \approx \eta_{-1}$ and $\eta(p_\perp/p) \approx \eta$ so that (22) becomes

$$\frac{\partial}{\partial\xi} < p_\perp >= -\lambda\xi\eta_{-1}. \quad (25)$$

In the McLerran-Venugopalan model used in Ref.1

$$\tilde{f}_s(t, \vec{p}) = \delta(p_z)\Theta(Q_s^2 - p_\perp^2)c\frac{N_c^2 - 1}{4\pi^3\alpha N_c} \quad (26)$$

immediately after the ion-ion collision leading to

$$\eta = c\frac{N_c^2 - 1}{4\pi^2\alpha N_c}Q_s^2. \quad (27)$$

While at early times

$$\eta_{-1} = c\frac{N_c^2 - 1}{2\pi^2\alpha N_c}Q_s \quad (28)$$

so that (25) becomes

$$\frac{\partial}{\partial\xi} < p_\perp >= -c\frac{\alpha N_c}{3\pi}Q_s\xi \quad (29)$$

Q_s is the saturation momentum. Defining

$$\bar{Q} = \frac{1}{\eta}\int d^2p \; p_\perp f_s = \frac{2}{3}Q_s$$

one finds

$$< p_\perp/\bar{Q} >= 1 - c\frac{\alpha N_c}{4\pi}\xi^2 \quad (30)$$

a behavior having an identical form, for ξ not too large, to that found in Ref.1.

Now turning to (19) it is clear that at early times the first term on the left-hand side of that equation and the second term on the right-hand side are small compared to the remaining two terms at early times so

$$< (p_z/\bar{Q})^2 >= c\frac{3\alpha N_c}{4\pi}\xi. \tag{31}$$

Over what range of ξ can (30) and (31) be used? Since we are working in a logarithmic approximation[7] we clearly need $\xi >> 1$ in order to take the collision term to be given by (11). In the McLerran-Venugopalan model $\xi \approx \ell n\ 1/\alpha$ even when t is as small as $1/Q_s$ so the assumption of large ξ should be (parametrically) good for all times after the gluons are freed in the heavy ion collision. In obtaining (25) from (22) we have used $\eta_{-1}^{\perp} \approx \eta_{-1}$ and $\eta(p_\perp/p) \approx \eta$. These assumptions are good so long as typical values of p_z are much less than typical values of $p_\perp$. We shall come back in a moment to determine when $|p_z| \approx p_\perp$. In solving (25) we have taken η_{-1} to be constant, and given by (28) in the McLerran-Venugopalan model. We can estimate the corrections coming from the time variation of η_{-1} by taking

$$\eta_{-1}(t) = \eta_{-1}\frac{\bar{Q}}{< p_\perp >}. \tag{32}$$

If we use (32) in (25) the result

$$< p_\perp/\bar{Q} >^2 = 1 - c\frac{\alpha N_c}{2\pi}\xi^2 \tag{33}$$

emerges. Eqs.33 and 30 are identical when $\alpha\xi^2 << 1$, however, we can expect (33) to be reasonable even when $\alpha\xi^2$ is not small. Thus, we suggest that (31) and (33) should be good estimates for the typical transverse and longitudinal momenta so long as $< p_z^2 >$ remains less than $< p_\perp >^2$. (The precise factor in front of the $\frac{\alpha N_c}{2\pi}\xi^2$ in (33) is not expected to be reliable, however, because (32) should be expected to be a reasonable estimate though not a precise evaluation.)

From (31) and (33) we see that $< p_z^2 >$ and $< p_\perp >^2$ become comparable when $\xi = \xi_1$ with ξ_1 given by

$$\xi_1 = \sqrt{\frac{2\pi}{c\alpha N_c}} + 0(\alpha^0). \tag{34}$$

When $\xi \geq \xi_1$ we can no longer expect (31) and (33) to be indicative of the evolution of the gluon system after the heavy ion collision. Eqs.(18), (19) and (22) should still be correct as they follow from (15), but when $\xi \geq \xi_1$ we are no longer able to estimate the terms appearing on the right-hand sides of (18), (19) and (22). While we are unable to follow the system, analytically, beyond $\xi = \xi_1$ we presume that typical values of $p_\perp$ and p_z remains equal, at $z = 0$, and that the gluonic system continues to evolve toward equilibrium.

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